RBE 2004

ระบบอัตโนมัติ (Automatic System)

สาขาวิศวกรรมหุ่นยนต์
คณะวิศวกรรมศาสตร์และเทคโนโลยีอุตสาหกรรม
มหาวิทยาลัยราชภัฏสวนสุนันทา

Chapter 2 Mathematical Model of Systems Lecture 3

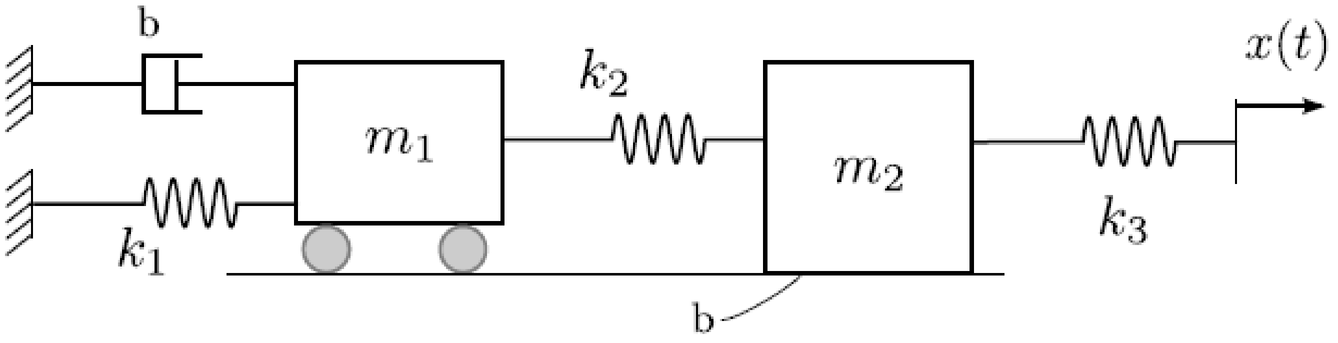
- การหา Transfer Function
  - ความสัมพันธ์ระหว่างอินพุทและเอาท์พุทของระบบพลศาสตร์, ฟังก์ชั่นการถ่ายโอน (Transfer Function)
- การทำ Linearization
- การหาค่าเอาท์พุ้ทซึ่งเป็นการที่ระบบตอบสนองต่ออินพุทชนิดต่างๆ
- Computer simulation (Matlab/Simulink)

Recap last lecture

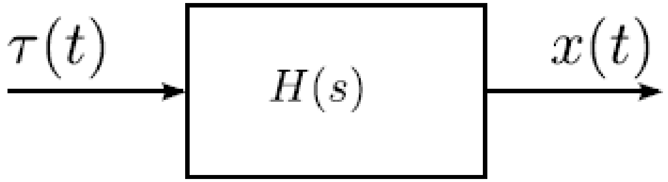
- System Modelling,
  - me,
  - ee,
  - miscellaneous
- Laplace transform table 2.3
- Final value theorem
- Spring-mass-damper model

Transfer Function of Linear System

If the spring is stretched to a point $x(t) = 5$ mm, held, then released at time $t = 0$, how does the position of m_1 evolve in time?



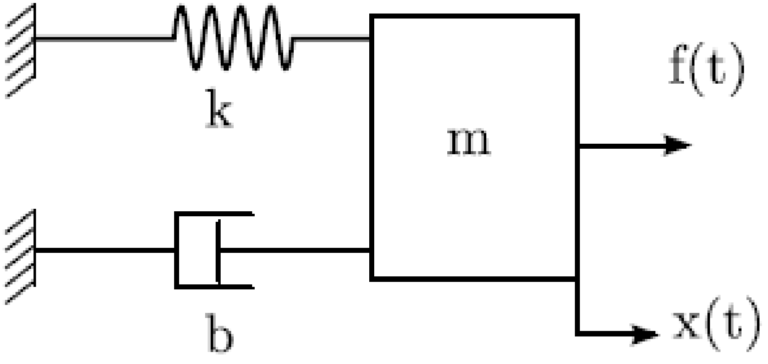
What torque must be applied to each of the robot joints so that end-effector moves along a given trajectory with a given speed?



Input/Output relation

Transfer function: A relation between the input and output of a linear system

Example: Input: force $f(t)$, output: displacement $x(t)$



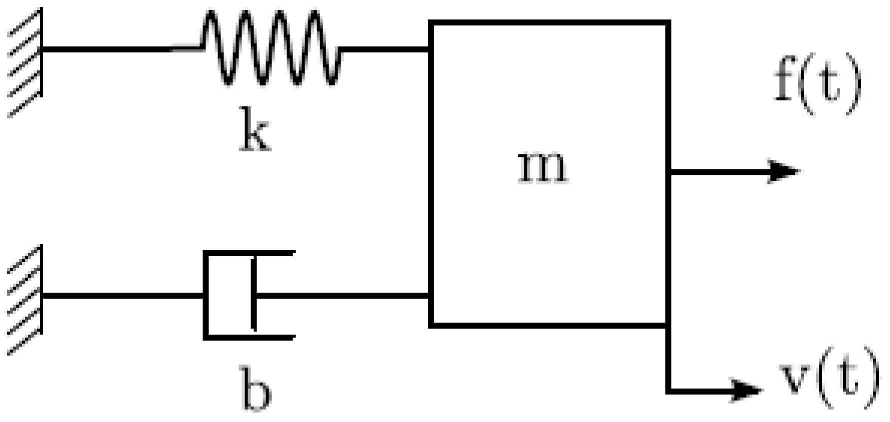
$$m\frac{d^2x}{dt} + b\frac{dx}{dt} + kx = f(t)$$

Taking the Laplace transform:

The transfer function is

$$H(s) = \frac{X(s)}{F(s)} = \qquad (1)$$

Example: Input: force $f(t)$, output: **velocity** $v(t)$



The dynamic model is

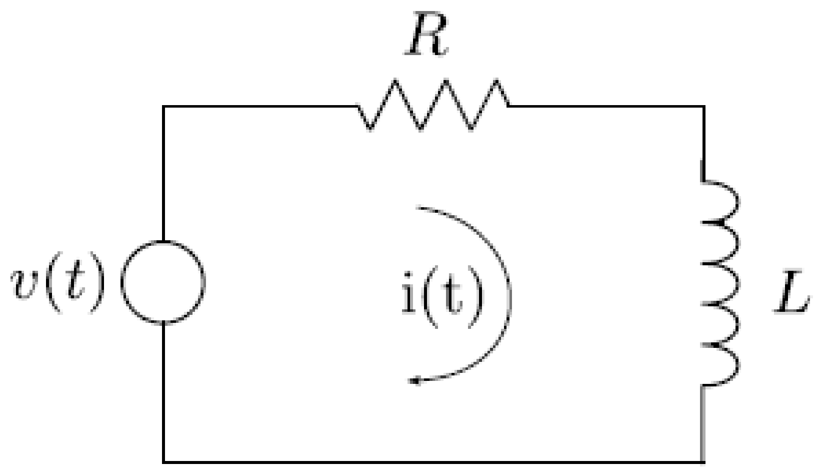
$$ \tag{2} $$

Laplace transform of (2) is

The transfer function is

$$H(s) = \frac{V(s)}{F(s)} = \tag{3}$$

Input: Voltage $v(t)$, output: Current $i(t)$



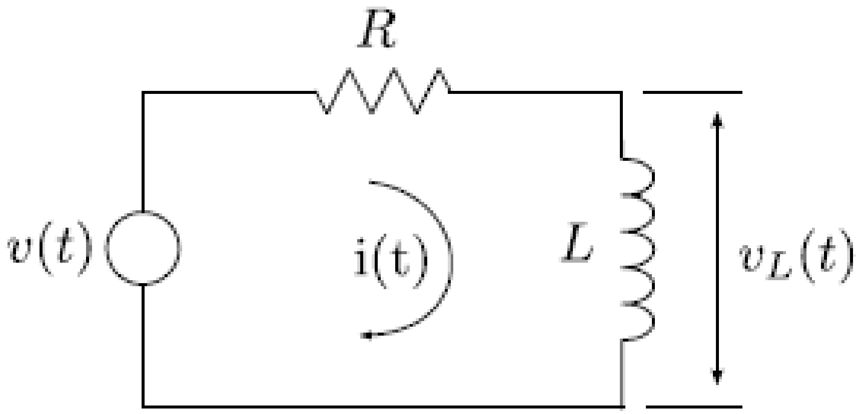
$$v(t) = Ri(t) + L\frac{di(t)}{dt} \tag{4}$$

Taking the Laplace transform of (4):

The transfer function is

$$H(s) = \frac{I(s)}{V(s)} =$$

Input: Voltage $v(t)$, **output**: $v_L(t)$



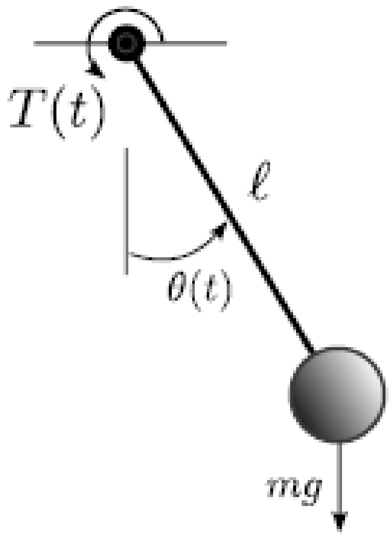
$$v(t) = Ri(t) + L\frac{di(t)}{dt}, \qquad v_L = L\frac{di(t)}{dt} \tag{5}$$

Taking the Laplace transform of (5):

The transfer function is

$$\frac{I(s)}{V(s)} = \frac{1}{Ls + R}, \qquad \frac{V_L(s)}{V(s)} =$$

The equation of motion of the simple pendulum is



$$m\frac{d^2\theta(t)}{dt^2} + m\frac{g}{\ell}\sin\theta = T(t) \quad (6)$$

For small angles, $\sin\theta \approx \theta$, in the frequency domain:

$$\frac{\theta(s)}{T(s)} = \quad (7)$$

Transfer function poles and zeros

Transfer function: A rational function in the complex variable $s = \sigma + j\omega$:

$$H(s) = \frac{N(s)}{D(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \ldots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0} \tag{8}$$

The zeros z_i are the roots of

$$N(s) = 0$$

Thus:

$$\lim_{s \to z_i} N(s) = 0 \tag{9}$$

The poles p_i are the roots of

$$D(s) = 0$$

Thus:

$$\lim_{s \to p_i} H(s) = \infty \tag{10}$$

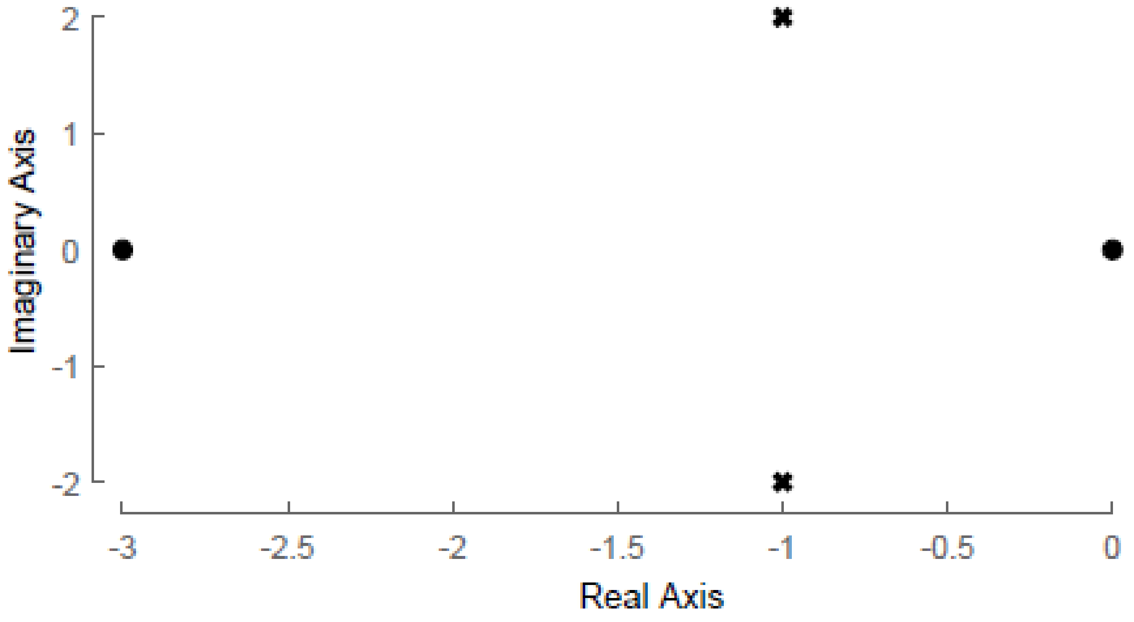
Poles and zeros

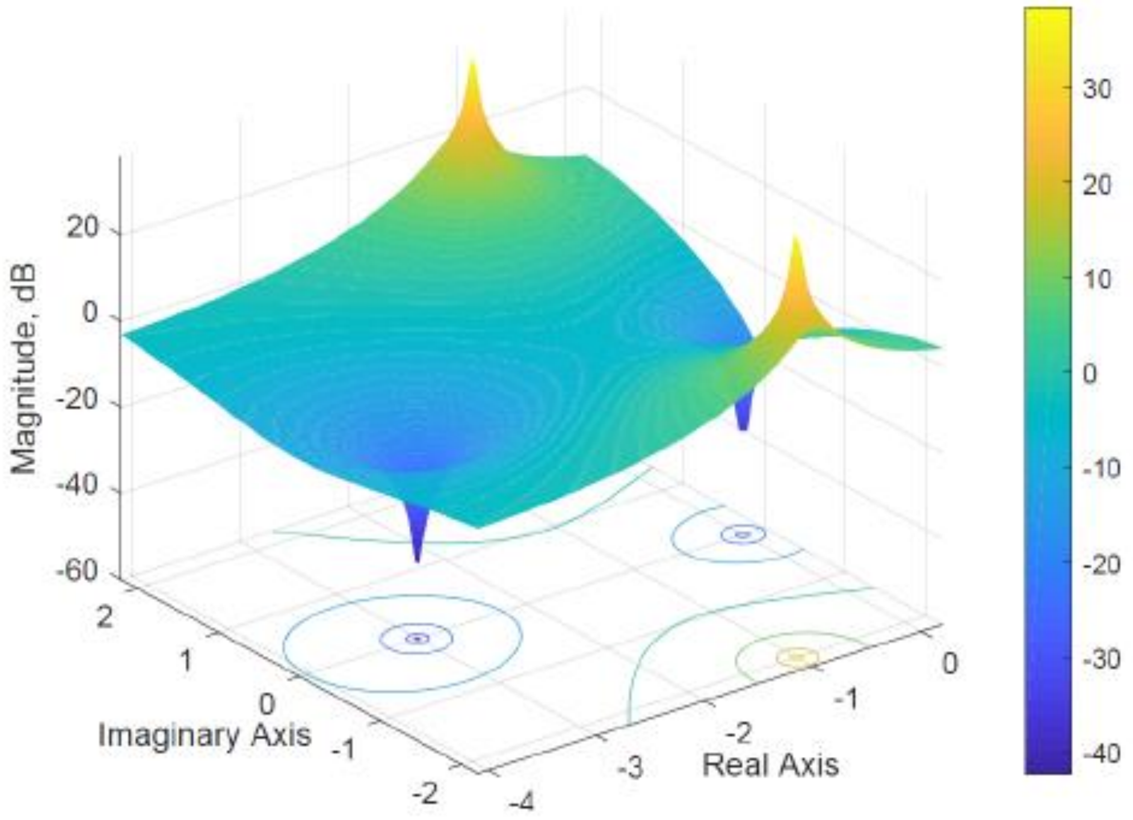
Consider the following function:

$$F(s) = \frac{s(s+3)}{s^2 + 2s + 5}$$

$\rightarrow$ Poles: $-1 + 2j,\ -1 - 2j$

$\rightarrow$ Zeros: $0,\ -3$





First order transfer functions

Order: The number of the highest derivative in the denominator (power of s)

Standard form of a first order system:

$$G(s) = k\frac{1}{\tau s + 1}$$

Characteristic equation: The denominator of the transfer function

$$H(s) = \frac{I(s)}{V(s)} = \frac{1}{Ls + R} =$$

Time constant: Characterizes the response to a step input of a first-order system.

$$\tau = \frac{L}{R} \tag{11}$$

$\rightarrow$ The denominator must be in the form of $\tau s + \mathbf{1}$

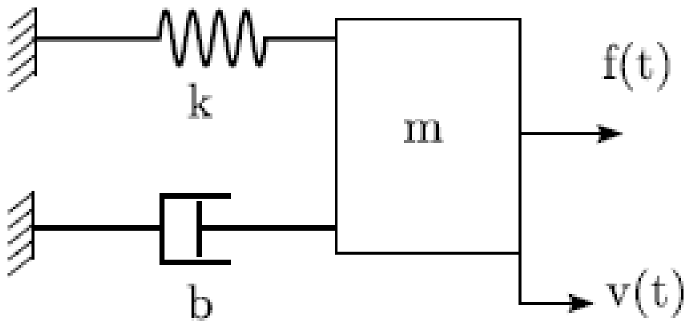
Second order transfer functions

Standard form

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{12}$$

Where: ω_n is the natural frequency, ζ is the damping ratio.

We will come back to these definitions in the next lecture.



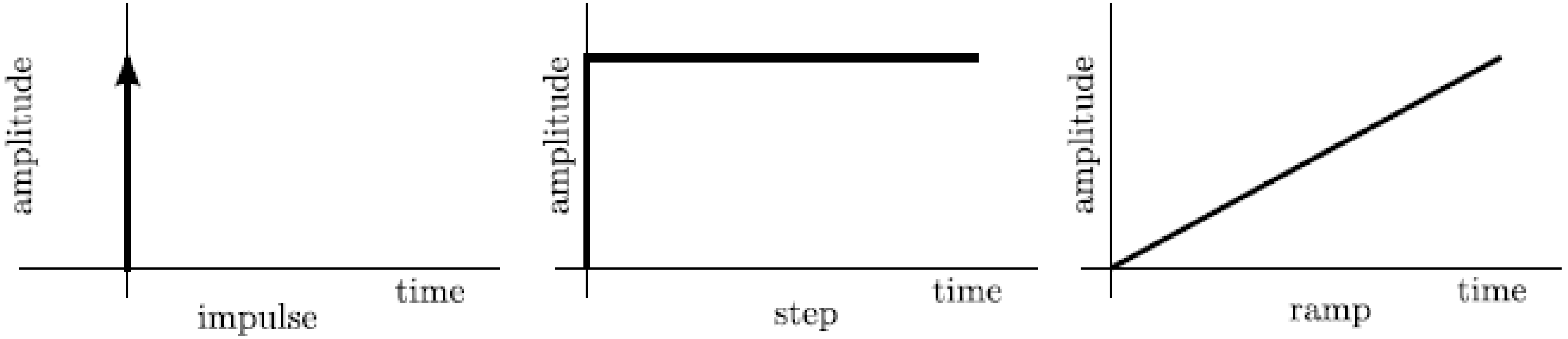
$$H(s) = \frac{X(s)}{F(s)} = \frac{1}{ms^2 + bs + k} =$$

$\Rightarrow \zeta = \frac{b}{2\sqrt{mk}}$: Dimensionless **damping ratio**

$\Rightarrow \omega_n = \sqrt{\frac{k}{m}}$: Natural frequency (rad/s)

System response

Step 1: Define the input signal in Laplace domain

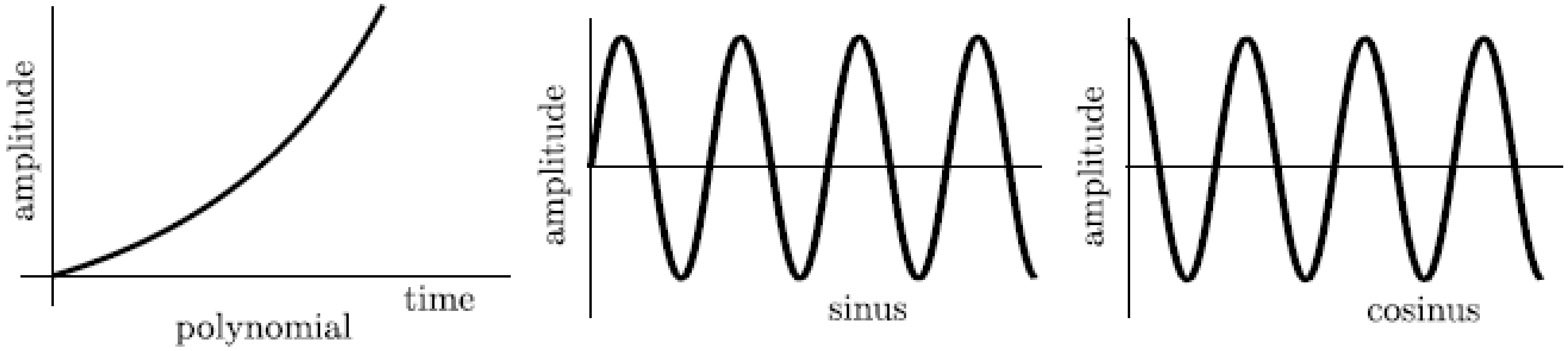


input signal	time domain	frequency domain
impulse	$r(t) = \delta(t)$	$R(s) = 1$
step	$r(t) = a$	$R(s) = a\frac{1}{s}$
ramp	$r(t) = at$	$R(t) = a\frac{1}{s^2}$

a is a constant

System response

Step 1: Define the input signal in Laplace domain



input signal	time domain	frequency domain
polynomial	$r(t) = at^n$	$R(s) = a\frac{n!}{s^{n+1}}$
sine	$r(t) = \sin(at)$	$R(s) = \frac{a}{s^2+a^2}$
cosine	$r(t) = \cos(at)$	$R(t) = \frac{s}{s^2+a^2}$

a is a constant

System response

Step 2: Replace the input signal in the transfer function

$$H(s) = \frac{X(s)}{F(s)} = \frac{1}{ms^2 + bs + k} \tag{13}$$

For a impulse input $f(t) = \delta(t)$, $F(s) = 1$ and the temporal response is

$$X(s) = \frac{1}{ms^2 + bs + k} \tag{14}$$

For a step-type input $f(t) = 1$ N, $F(s) = 1/s$ and the temporal response is

$$X(s) = \frac{1}{ms^2 + bs + k}\left(\frac{1}{s}\right) \tag{15}$$

For a sinusoidal input $f(t) = 5\sin(t)$ N:

$$X(s) = \frac{1}{ms^2 + bs + k}\left(\frac{5}{s^2 + 1}\right) \tag{16}$$

System response

Step 3: Calculate the inverse transform of the resulting function

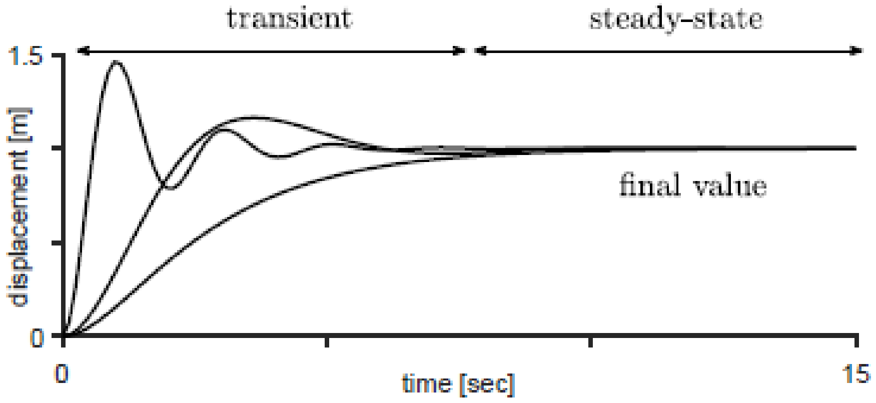
For a impulse input $f(t) = \delta(t)$, $F(s) = 1$ and the temporal response is

$$x(t) = \mathscr{L}^{-1}\left\{\frac{1}{ms^2 + bs + k}\right\} \tag{17}$$

For a step-type input $f(t) = 1$ N, $F(s) = 1/s$ and the temporal response is

$$x(t) = \mathscr{L}^{-1}\left\{\frac{1}{ms^2 + bs + k}\left(\frac{1}{s}\right)\right\} \tag{18}$$

and so on.



Steady-state value

Final value theorem: Gives the steady-state value without computing the inverse transform.

If the function converges, i.e., the poles of $sX(s)$ have negative real parts, then:

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s) \tag{19}$$

For a step type input, the mass spring damper system settles at

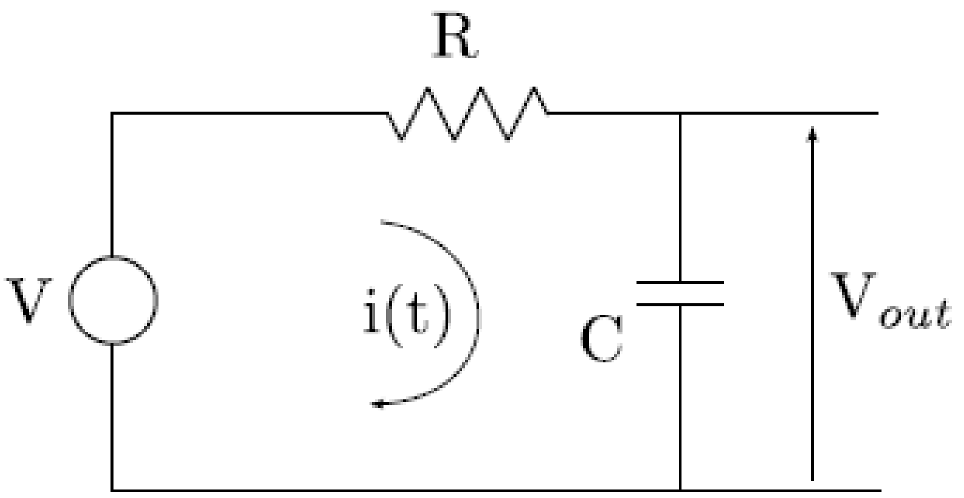
$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s) = \lim_{s\to 0} s\left\{\frac{1}{ms^2+bs+k}\left(\frac{1}{s}\right)\right\} = \tag{20}$$

For a impulse input, the RL system converges at

$$\lim_{t\to\infty} i(t) = \lim_{s\to 0} sI(s) = \lim_{s\to 0} s\left\{\frac{1}{Ls+R}\right\} = \tag{21}$$

Exercise 16

Find the transfer function $H(s)$ between the input voltage V and the output voltage V_{out}.



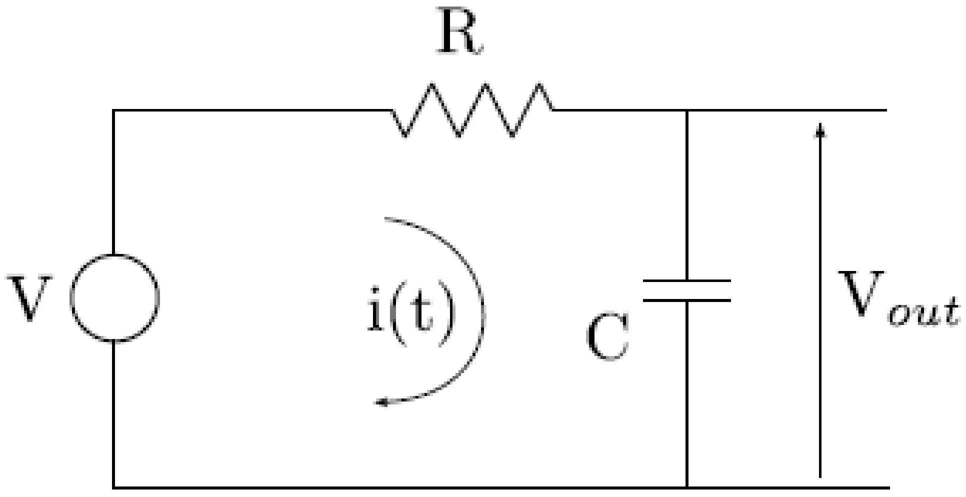
Procedure:

→ Find the differential equation for the current

→ Find the equation for the output voltage

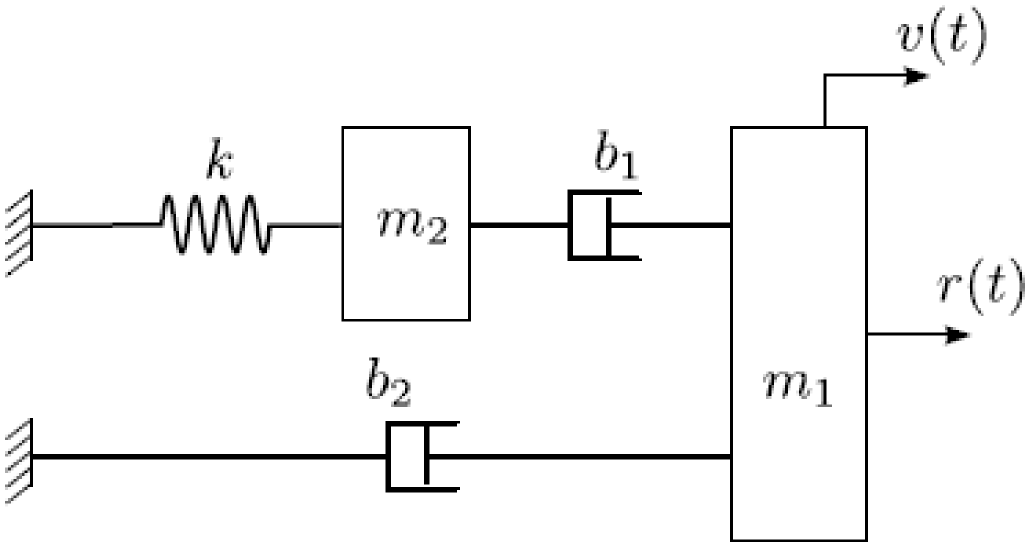
→ Calculate the transfer function

Exercise 16 - continued



Exercise 17

Find the transfer function $H(s) = \frac{V(s)}{R(s)}$ between the force $r(t)$ and the velocity of mass m_1.



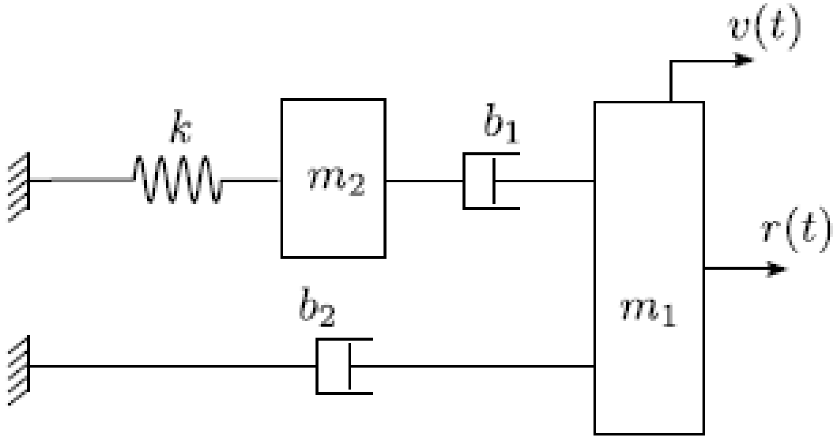
Procedure:

→ Find the differential equation the velocity of each mass

→ Calculate the Laplace transform

→ Calculate the transfer function

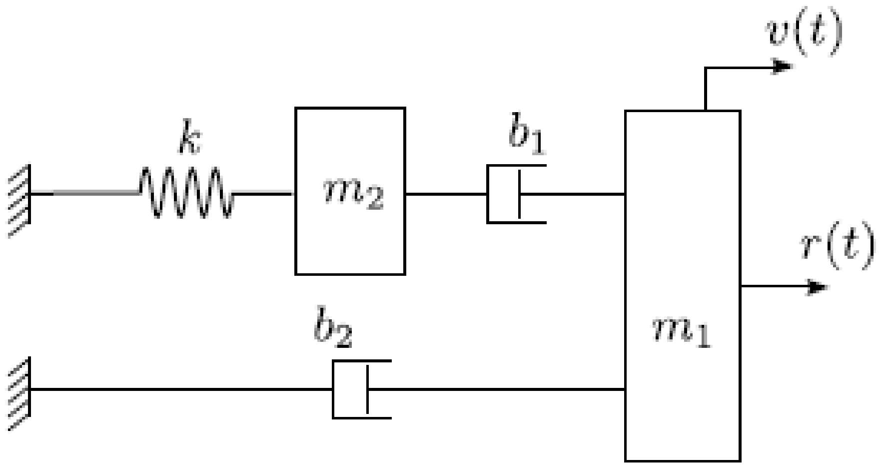
Exercise 17 - continued



Exercise 17 - continued

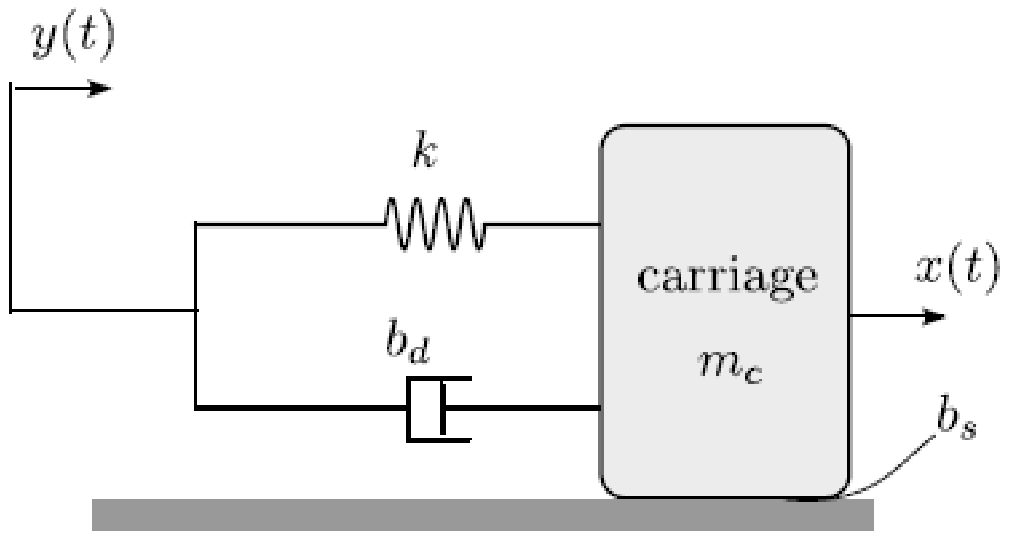
$$[m_1 s + (b_1 + b_2)]V(s) - b_1 V_2(s) = R(s)$$

$$-b_1 V(s) + \left(m_2 s + b_1 + \frac{k}{s}\right) V_2(s) = 0$$



Exercise 18

A high precision positioning slide is shown in the figure. The drive shaft friction is $b_d = 0.65$, the drive shaft spring constant is $k = 1.8$, $m_c = 1$, and the slide friction is $b_s = 0.9$.



Determine:

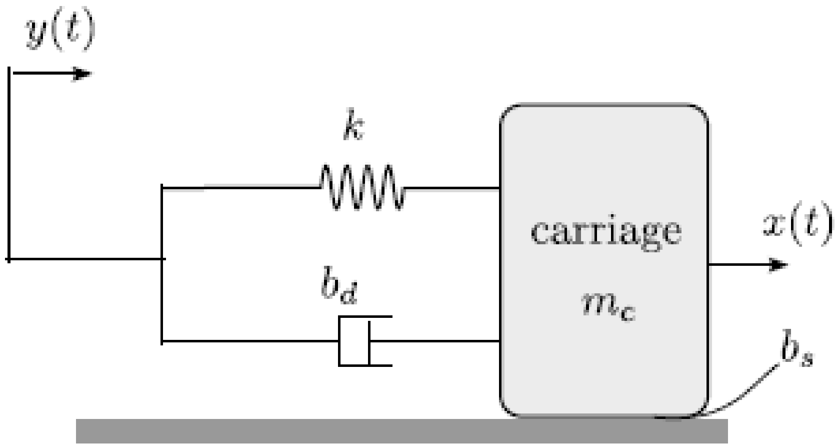
→ Find the transfer function $H(s) = X(s)/Y(s)$.

→ Calculate the natural frequency, damping ratio, the poles, and zeros of $H(s)$

→ Find the steady-state value for a step input

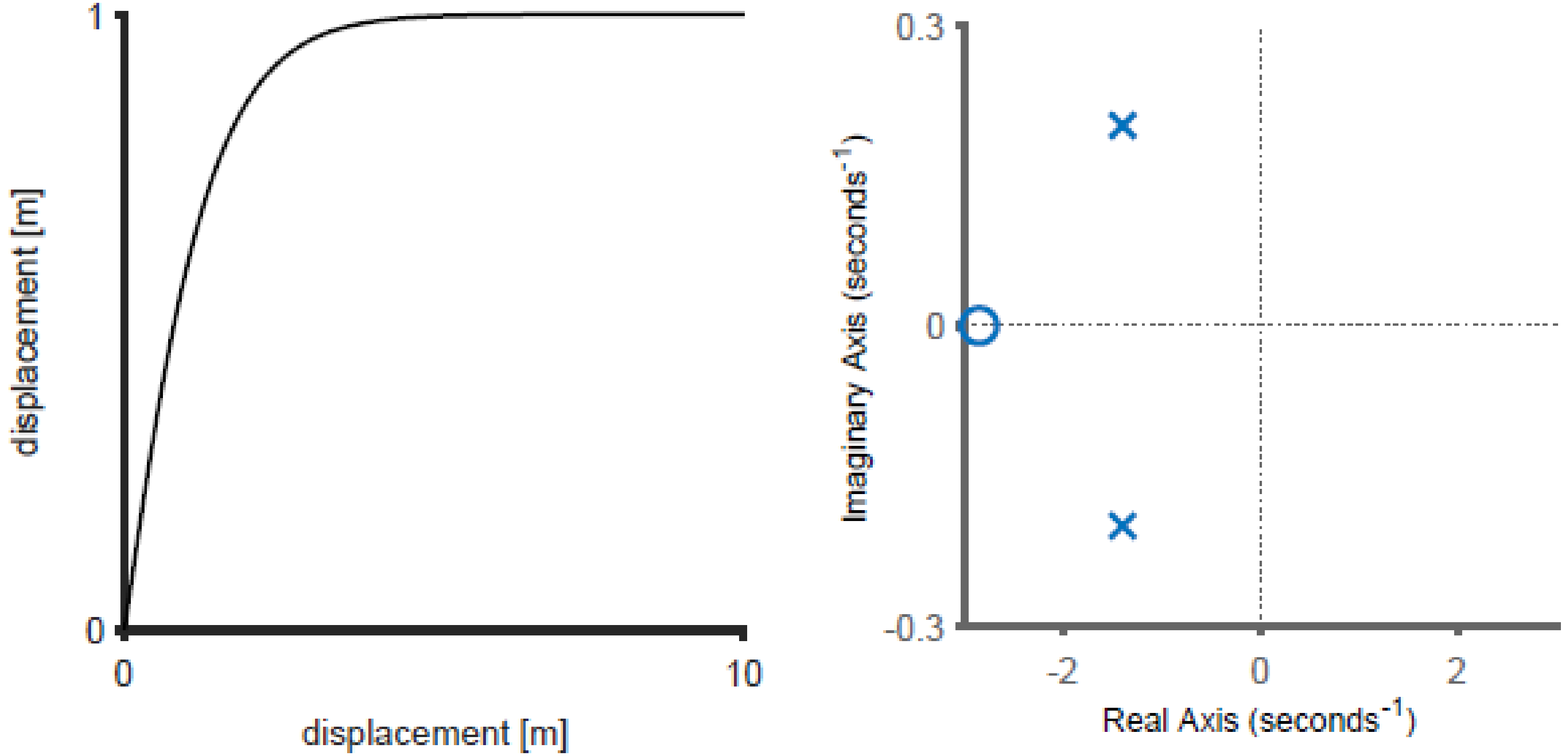
→ Plot the step response of using Matlab

Exercise 18 - continued



Exercise 18 - continued - Using Matlab

H = tf([0.7 2],[1 2.8 2]) → Transfer function
damp(H) → Natural frequency and damping
step(H,10) → Step response
pzplot(H) → Location of zeros and poles



Exercise 19

Calculate the natural frequency and damping ratio of the following transfer function

$$T(s) = \frac{1.05 \times 10^7}{2s^2 + 1.6 \times 10^3 s + 1.05 \times 10^7}$$

Determine:

→ Write the transfer function in standard form

→ Find the steady-state value for a step input

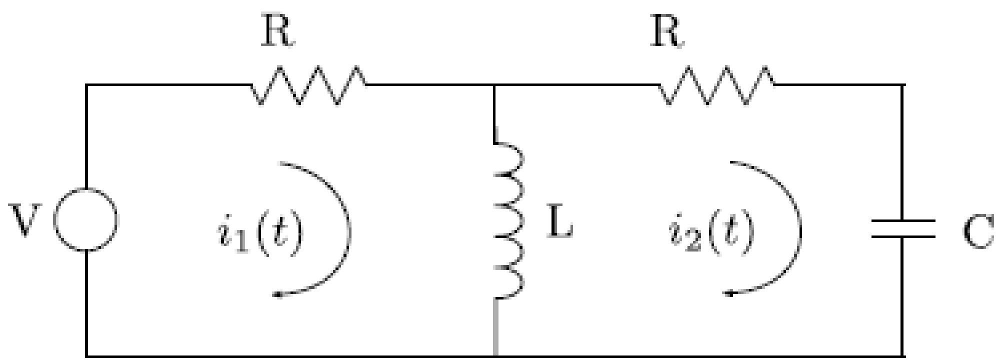
→ Calculate the natural frequency and damping ratio

Exercise 19 - continued

$$T(s) = \frac{1.05 \times 10^7}{2s^2 + 1.6 \times 10^3 s + 1.05 \times 10^7}$$

Exercise 20

Find the transfer function $G(s) = \frac{I_2(s)}{V(s)}$ of the circuit shown. Then, calculate the step response of the circuit using Matlab. Take $R = 10\ \Omega$, $C = 0.001$ F, $L = 0.1$ H, $V = 5$ V.



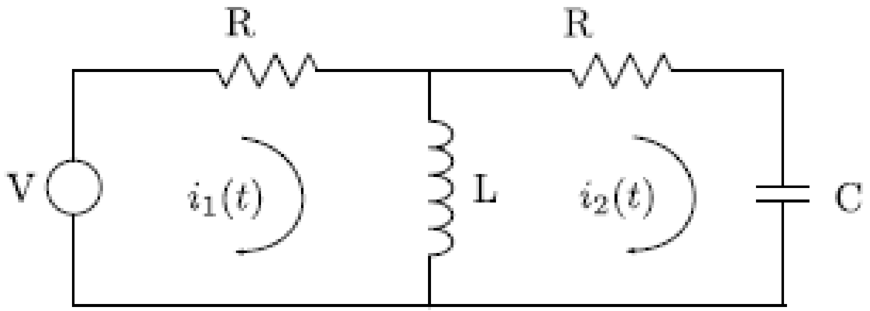
Determine:

→ Find the transfer function $H(s) = I_2(s)/V(s)$.

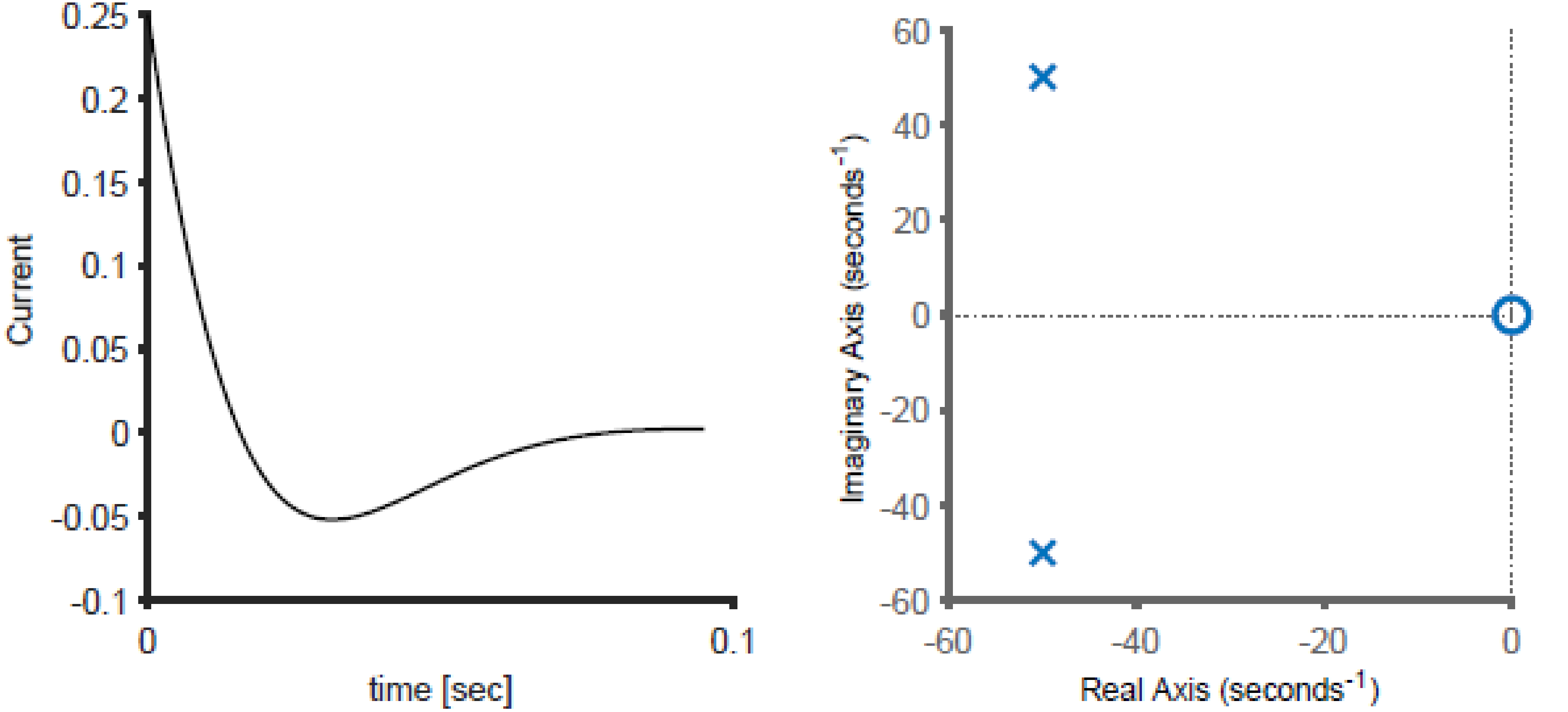
→ Find the steady-state value for a step input

→ Plot the step response of using Matlab

Exercise 20 - continued



Exercise 20 - continued - Using Matlab



Approximation of Linear System

Definition : a system is defined as linear in term of input and output response if it satisfies the properties of superposition and homogeneity.

Necessary condition :

- Superposition:

  Assume given x1 to a system that yield y1

  given x2 to a system that yield y2

  Thus, for linear system,

  given x1+x2 to a system that yield y1+y2

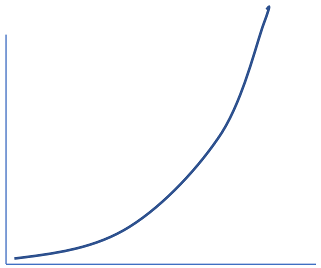
- Homogeneity:

  if there is a scale factor β

  given βx1 to a system that yield βy1

Not satisfying superposition and homogeneity

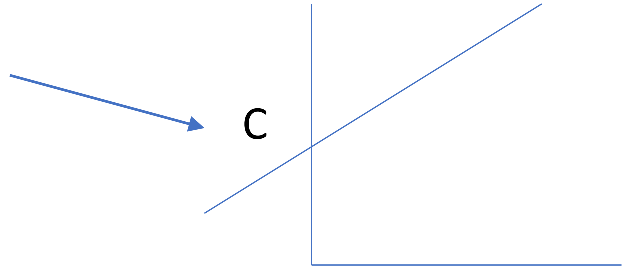
$$y = x^2$$



Not linear

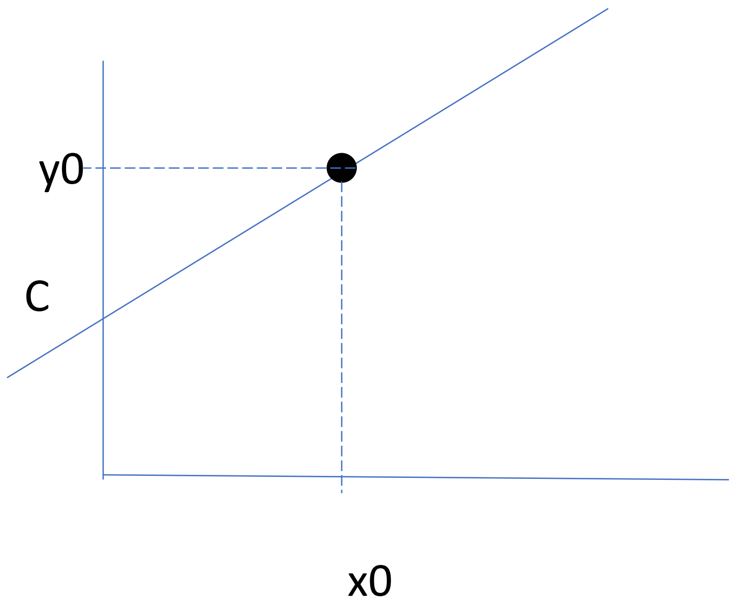
$$y = mx + \boxed{C}$$

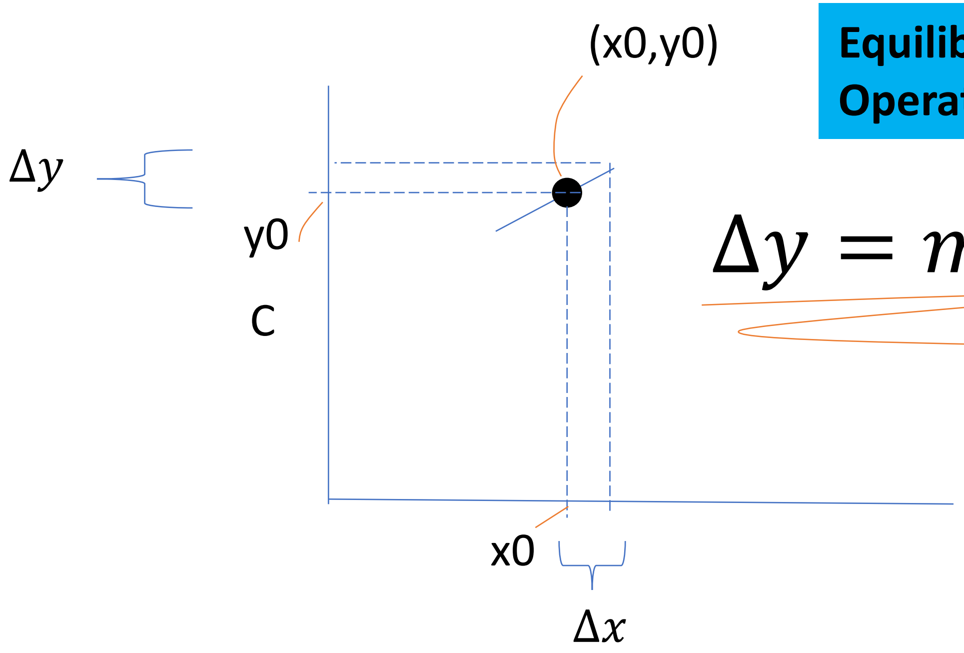
Not linear



How to solve this problem???

Considering an operating point x0, y0, for small change Δx and Δy





Equilibrium point/ Operating point

$$\Delta y = m\Delta x$$

$$y = mx + C$$

$$y_0 + \Delta y = m(x_0 + \Delta x) + C$$

$$y_0 + \Delta y = mx_0 + C + m\Delta x$$

Known that $y_0 = mx_0 + C$

Taylor series approximation

Using Taylor series expansion about the operating point (x0,y0) for a small range, we can obtain linear approximation of a nonlinear system.

The relationship of the two variables is written as

$$y(t) = g(x(t)), \tag{2.6}$$

Taylor series expansion about the operating point may be utilized [7]. Then we have

$$y = g(x) = g(x_0) + \left.\frac{dg}{dx}\right|_{x=x_0} \frac{(x - x_0)}{1!} + \left.\frac{d^2g}{dx^2}\right|_{x=x_0} \frac{(x - x_0)^2}{2!} + \cdots. \tag{2.7}$$

The slope at the operating point,

$$\left.\frac{dg}{dx}\right|_{x=x_0},$$

The slope at the operating point,

$$\left.\frac{dg}{dx}\right|_{x=x_0},$$

is a good approximation to the curve over a small range of $(x - x_0)$, the deviation from the operating point. Then, as a reasonable approximation, Equation (2.7) becomes

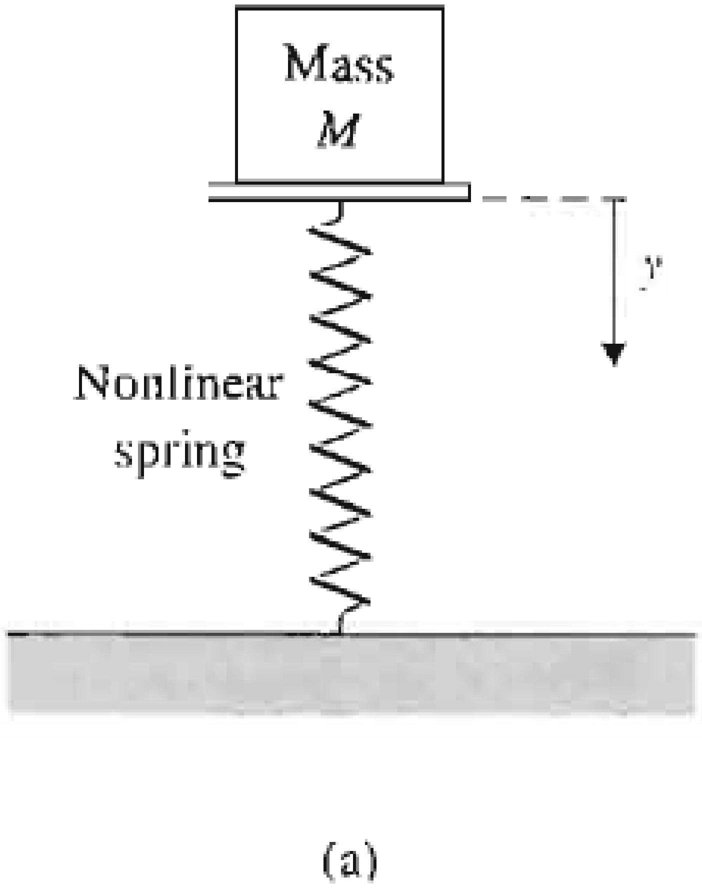
$$y = g(x_0) + \left.\frac{dg}{dx}\right|_{x=x_0}(x - x_0) = y_0 + m(x - x_0), \tag{2.8}$$

where m is the slope at the operating point. Finally, Equation (2.8) can be rewritten as the linear equation

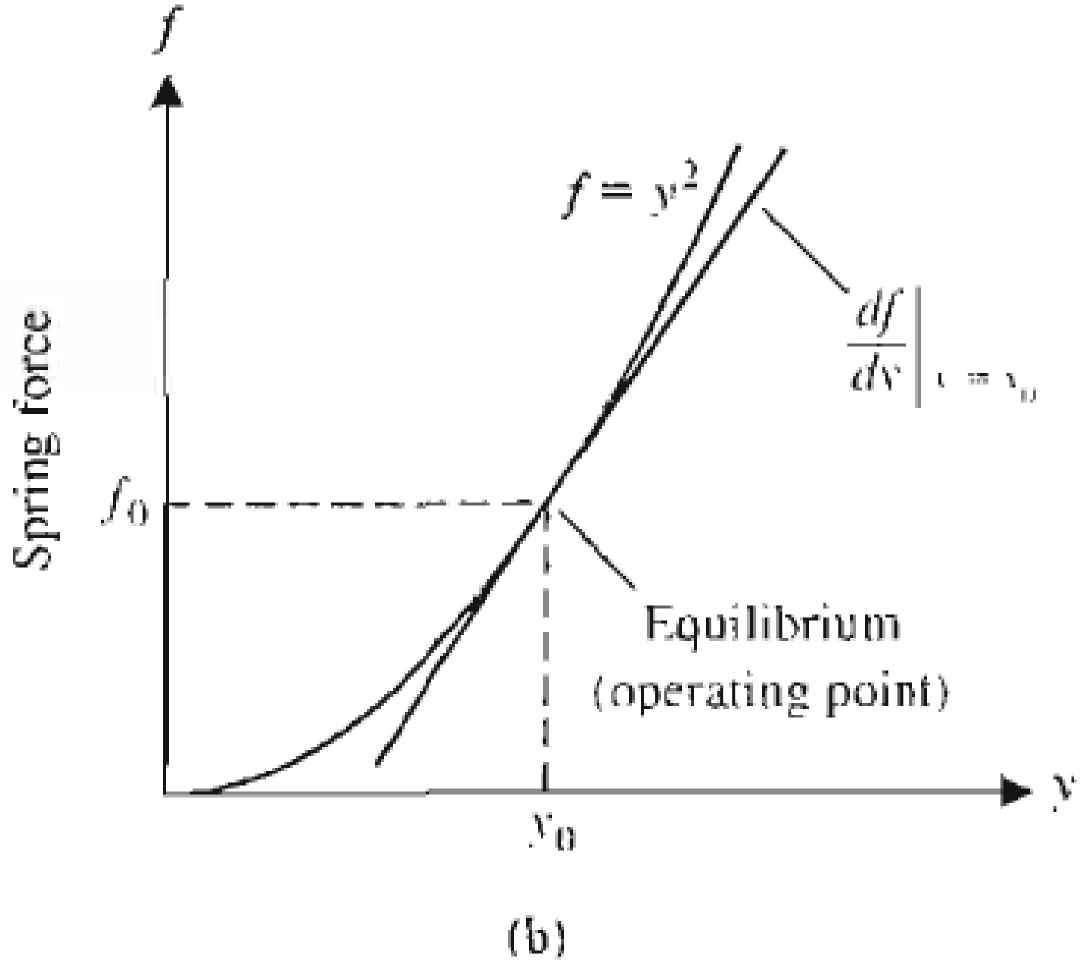
$$(y - y_0) = m(x - x_0)$$

or

$$\Delta y = m\,\Delta x. \tag{2.9}$$



Mass
M
y
Nonlinear
spring
(a)



f
$f = y^2$
Spring force
f_0
Equilibrium
(operating point)
y
y_0
(b)

Computer Simulation

- Matlab/Simulink

Example